Math 246C Lecture 28 Notes

Daniel Raban

May 31, 2019

1 L^2-Estimates for the $\overline{\partial}$-Operator

1.1 Solution of the $\overline{\partial}$ problem

Recall that

$$\sum_{j,k=1}^n \int_\Omega \frac{\partial\varphi}{\partial z_j \partial \overline{z}_k} f_j \overline{f}_k e^{-\varphi}\, L(dz) \leq 2\int |f|^2 |\partial\psi|^2 e^{-\varphi} + 2\|T^* f\|_{\varphi_1}^2 + \|Sf\|_{\varphi_3}^2,$$

where $f \in C^\infty_{0,(0,1)}(\Omega)$, $\Omega \subseteq \mathbb{C}^n$ is open, $\varphi_1 = \varphi - 2\psi$, and $\varphi_3 = \varphi$. Assume that $\varphi \in C^\infty(\Omega)$ is **strictly plurisubharmonic**: there exists $0 < c(z) \in C(\Omega)$ such that

$$\sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq c(z)|w|^2, \qquad z \in \Omega, w \in \mathbb{C}^n.$$

First consider the simplest case, $\Omega = \mathbb{C}^n$. We can then take $\psi = 0$, and it follows that

$$\int c|f|^2 e^{-\varphi} \leq \|T^* f\|_\varphi^2 + \|Sf\|_\varphi, \qquad f \in C^\infty_{0,(0,1)}(\mathbb{C}^n).$$

Recall that $T = \overline{\partial} : L^2(\mathbb{C}^n, e^{-\varphi}) \to L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ and $S = \overline{\partial} : L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi}) \to L^2_{(0,2)}(\mathbb{C}^n, e^{-\varphi})$ are closed and densely defined with natural domains, By the density lemma, this inequality extends to all $f \in D(T^*) \cap D(S)$.

Theorem 1.1. *Let $\varphi \in C^\infty(\mathbb{C}^n)$ be strictly plurisubharmonic with*

$$\sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq c(z)|w|^2, \qquad 0 < c \in C(\mathbb{C}^n).$$

Then for all $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ with $\partial g = 0$ and $\int |g|^2/ce^{-\varphi} < \infty$, there exists some $u \in L^2(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{\partial} u = g$ and

$$\int |u|^2 e^{-\varphi} \leq \int \frac{|g|^2}{c}.$$

Proof. We must solve the equation $Tu = g$ so that the above conclusion holds. Note that

$$\begin{aligned} Tu = g &\iff \forall f \in D(T^*), \langle Tu, f\rangle_\varphi = \langle g, f\rangle \qquad (D(T^*) \text{ is dense}) \\ &\iff \langle u, T^* f\rangle_\varphi = \langle g, f\rangle_\varphi \ \forall f \in D(T^*) \qquad (T \text{ is closed}). \end{aligned}$$

We claim that

$$|\langle g, f\rangle_\varphi| \le \|T^* f\|_\varphi \left(\int \frac{|g|^2}{c} e^{-\varphi}\right)^{1/2}, \qquad f \in D(T^*).$$

Indeed, if f is orthogonal to $\ker(S) \ni g$, then the left hand side equals 0. Also, $\operatorname{ran}(T) \subseteq \ker(S)$, so if $\langle f, Tu\rangle_\varphi = 0$ for all $u \in D(T)$, then $f \in D(T^*)$ and $T^* f = 0$; so the right hand side equals 0. If $f \in D(T*) \cap \ker(S)$, we get (by Cauchy-Schwarz) that

$$\begin{aligned} |\langle g, f\rangle_\varphi|^2 &= \left|\int \langle g, \overline{f}\rangle e^{-\varphi}\right|^2 \\ &\le \left(\int c|f|^2 e^{-\varphi}\right) \int \frac{|g|^2}{c} e^{-\varphi} \\ &\le \|T^* f\|_\varphi^2 \int \frac{|g|^2}{c} e^{-\varphi}. \end{aligned}$$

The claim follows, and the antilinear form $T^* f \mapsto \langle g, f\rangle_\varphi$ for $f \in D(T^*)$ extends to a continuous linear form on $L^2(\mathbb{C}^n, e^{-\varphi})$ with norm $\le \left(\int \frac{|g|^2}{c} e^{-\varphi}\right)^{1/2}$.

So there exists some $u \in L^2(\mathbb{C}^n, e^{-\varphi})$ with $\|u\|_\varphi^2 \le \int \frac{|g|^2}{c} e^{-\varphi}$ and $\langle g, f\rangle_\varphi = \langle u, T^* f\rangle$ for all $f \in D(T^*)$. So $u \in D(T)$, and $Tu = g$. □

1.2 Extensions

Arguing as in the 1 dimensional case, replacing φ by $\varphi + 2\log(1 + |z|^2)$ (the latter term is strictly plurisubharmonic on $\mathbb{C}^n$) and regularizing φ, we get the following result:

Theorem 1.2. *Let $\varphi \in \mathrm{PSH}(\mathbb{C}^n)$ with $\varphi \not\equiv -\infty$. For all $g \in L^2_{(0,1)}(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{g} = 0$, there exists a $u \in L^2_{\mathrm{loc}}(\mathbb{C}^n, e^{-\varphi})$ such that $\overline{\partial} u = g$ and*

$$2\int |u|^2 e^{-\varphi}(1 + |z|^2)^{-2} \le \int |g|^2 e^{-\varphi}.$$

Remark 1.1. There exist analogous results when $\mathbb{C}^n$ is replaced by an open set $\Omega \subseteq \mathbb{C}^n$, provided that Ω is **pseudoconvex**: there exists $u \in C(\Omega) \cap \mathrm{PSH}(\Omega)$ such that for all $t \in \mathbb{R}$, the set $\{z \in \Omega : u(z) < t\}$ is relatively compact in Ω. (Notice that any open set $\Omega \subseteq \mathbb{C}$ is pseudoconvex.)